Order by Disordered Action in Swarms

Gerardo Beni
University of California Riverside

Goal

To show that
for *swarms*
lack of synchronicity
is not a hindrance
but an advantage

System considered

- ~ 10^2-$10^{<<23}$ units
- ~ identical
- ~ short-range
- No centralization No common clock

What these systems are expected to do

- Tasks beyond the capability of a single unit
- Emergent collective ‘bio-like’ behavior

Collective bio-behavior

- Self-organization
- Growth
- Development
- Reproduction
- Adaptation
- Learning

ultimately depend on pattern formation

Engineering problem

- Designing systems capable of carrying out Collective bio-behavior (CBB) tasks
  - Many difficulties
  - One common difficulty in most CBB tasks:

  engineering a method of pattern formation

Engineering pattern formation

- Several methods for CBB design have been proposed
- Most methods are imported from Physics and/or Computer Science

Some methods for engineering pattern formation

- Force fields
- Non-linear dynamics
- PDE
- Diffusion-reaction models
- Cellular automata
- Evolutionary algorithms
- Stochastic processes

All* methods listed previously share one basic feature:

they evolve according to a common clock

- Explicitly
- Implicitly (as in most centralized controls)

(*) There are always some exceptions

Bio systems have no(*) common clock

But, the most prevalent modes of evolution of engineered systems are:

- Parallel (synchronous)
- Sequential (asynchronous)

require a common clock

(*) There are always some exceptions

Swarm model based on difference equations

- Advantages
  - Quantitative
  - Predictable

- Challenge
  - Swarms are ‘decentralized’ and ‘not synchronous’
  - Difference equations are ‘centralized and synchronous’

Difference equations from differential equations

- Example: diffusion equation

$$\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2}$$

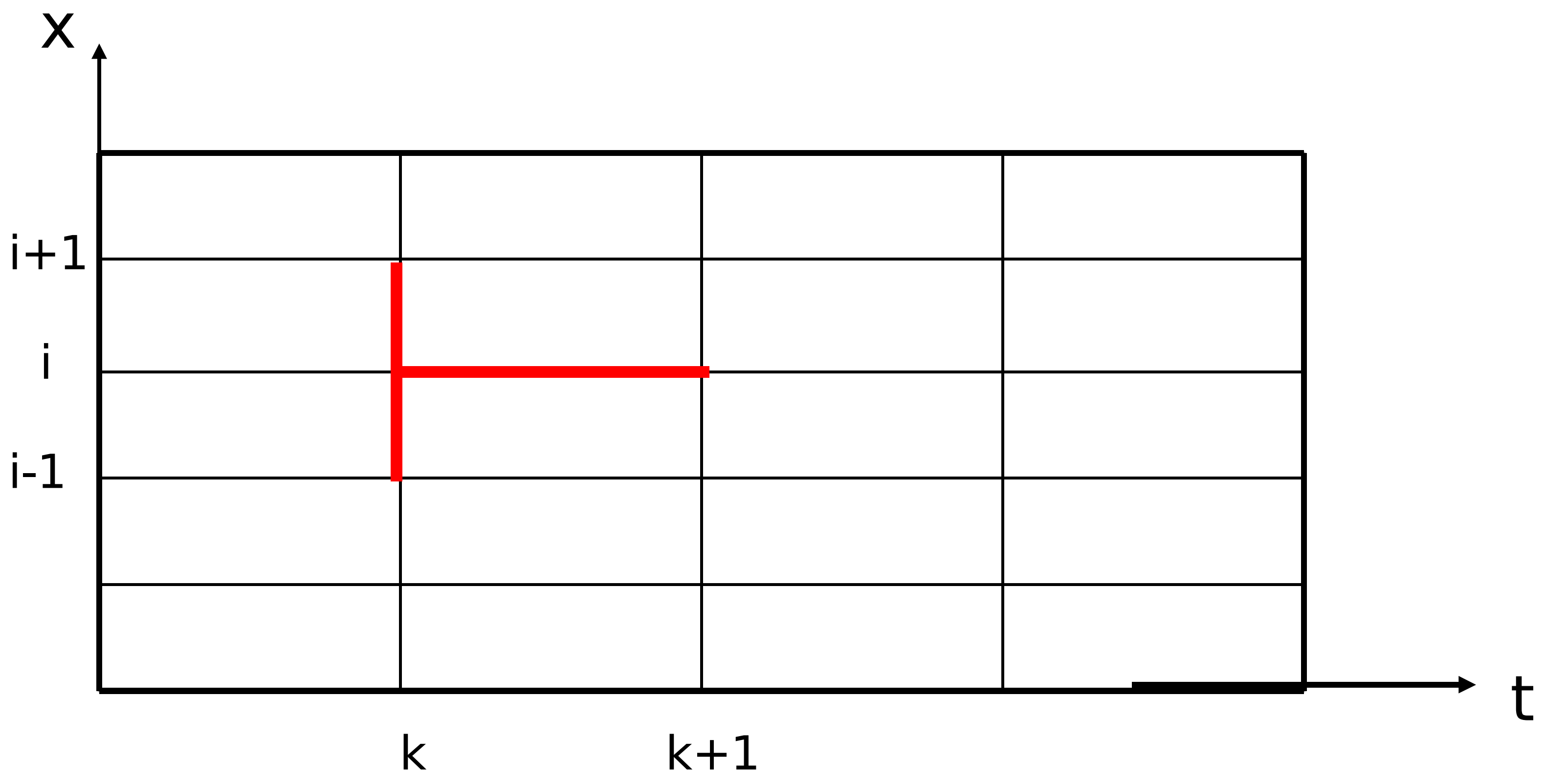
Diffusion difference eqn.

$$u_i^{(k+1)} - u_i^{(k)} = K\alpha\left(u_{i+1}^{(k)} - 2u_i^{(k)} + u_{i-1}^{(k)}\right)$$

$$\alpha \equiv \frac{\delta t}{(\delta x)^2}$$

= Time step/(spatial step)^2

Solution by forward evolution



Note: Synchronicity in x   common clock updates

Generalization: difference eqns. maps

- map is a more general concept
- A map is a formula that describes a new state in terms of the previous state.
- Most current research using maps focuses on non-linear maps (attractors, chaos,etc.)

Linear maps are also interesting

$$u^{(k+1)} = Hu^{(k)} + v$$

u, v are N-dim vectors and H is a NxN matrix

Definition of Swarm as a map

- Vague definition:
  - Swarm is some sort of self-organizing entity
- Less vague definition:
  - Swarm is a *self-updating* map
- Math definition:

  next slide

Swarm(*) definition

- **A vector S such that any of its component** (i=1…N) **updates S on local time t_i according to a function s_i of S**
- s_i is often more conveniently written as a function u_i of a function $v_i(S,t_i)$: $s_i = u_i\ (v_i\ (S,t_i))$
- In this way we can separate the function that singles out the components to update from the mode of updating them
- An example is the case of v_i being linear, and u_i being non linear or, e.g., stochastic.

(*) not intended to establish what a “swarm” is; this is simply an operational de

Compare Swarms with a well known case of maps:

The numerical solutions of linear systems

using <u>iterative methods</u>

Numerical solution of linear systems

System to solve

$$\boxed{Au = b}$$

Iterative solution of this system

$$\boxed{u^{(k+1)} = u^{(k)} + W^{-1}\left(b - Au^{(k)}\right)}$$

NOTE: algorithms differ by the choice of the ‘conditioning matrix’ W

Classic iterative methods

- Jacobi

$$W = D$$

- Gauss-Seidel

$$W = D + L$$

- Jacobi over-relaxation JOR

$$W = \omega^{-1} D$$

- Successive over-relaxation SOR

$$W = \omega^{-1} D + L$$

Convergence

- Iterative schemes are generally recast as

$$u^{(k+1)} = Hu^{(k)} + W^{-1}b$$

$$H = I - W^{-1}A$$

- Convergence depends only on H
- A necessary and sufficient condition is :

spectral radius of H $\rho(H) < 1$

Role of spectral radius

- Spectral radius $$\rho(H)$$

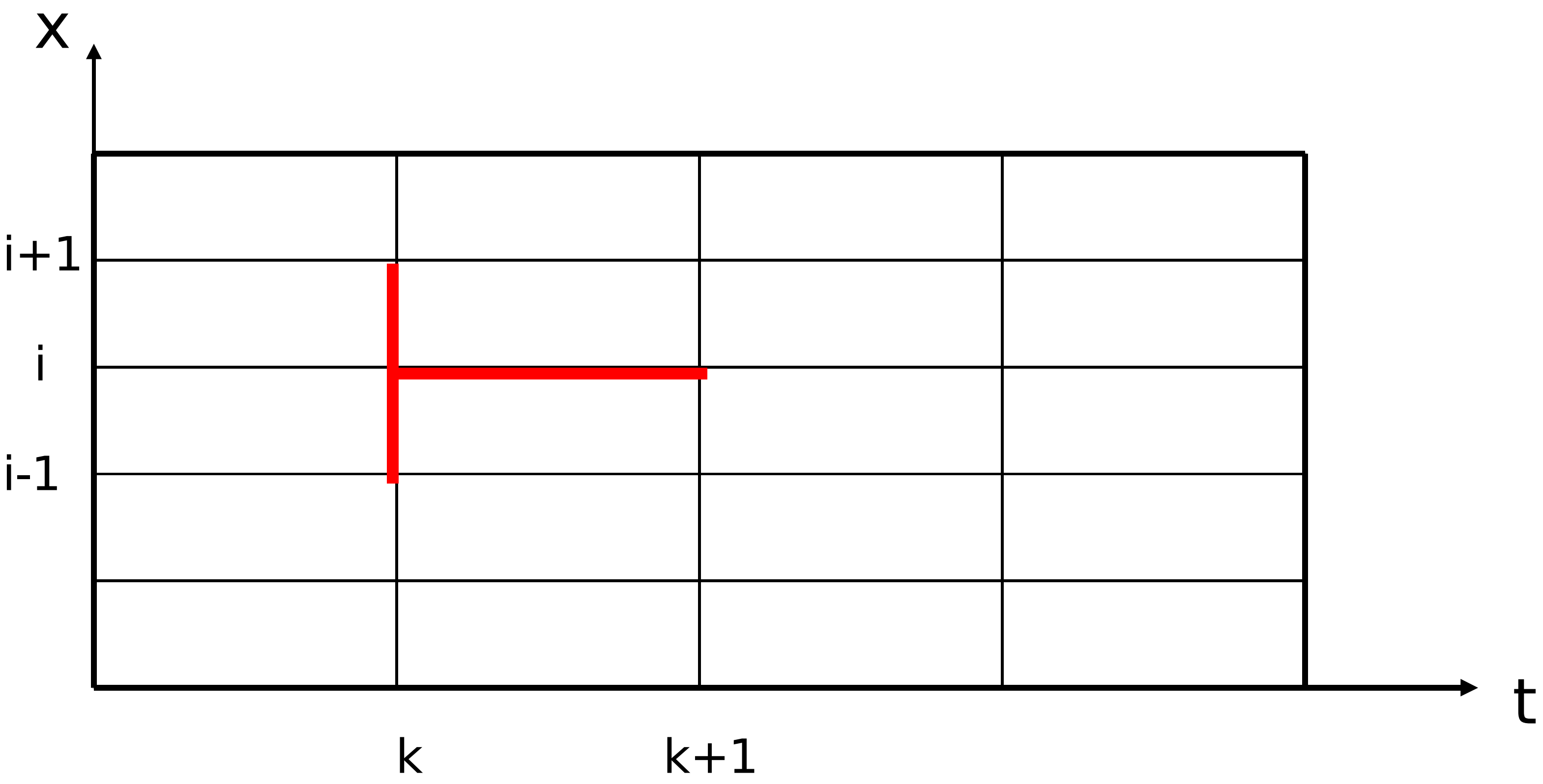
  (max modulus of eigenvectors)

$$\rho(H) < 1$$

- Convergence *iff*

- Asymptotic rate of convergence $$R_{\infty} = -\ln \rho(H)$$

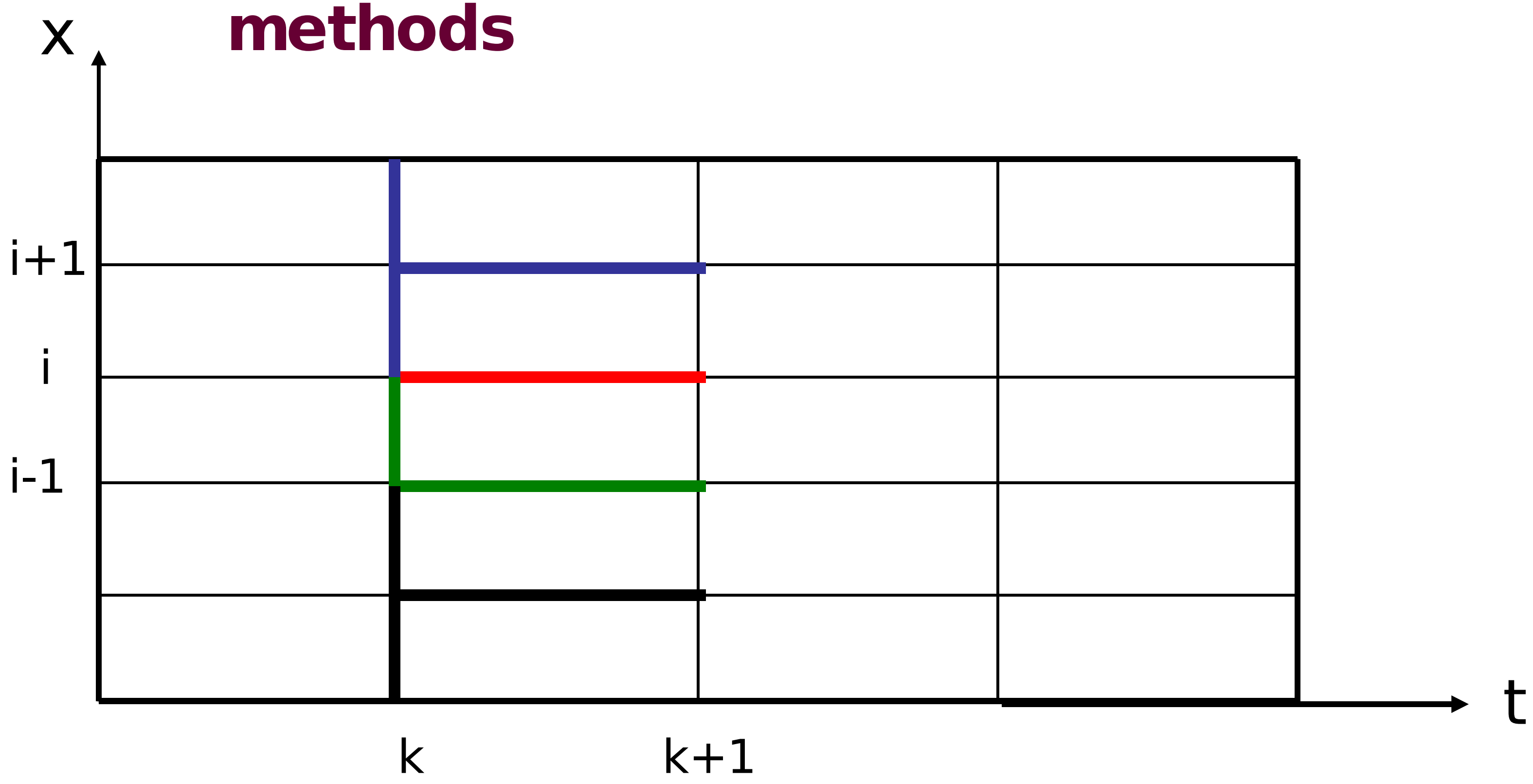
Forward evolution



Note: Synchronicity in x and ‘centralized control’ updates

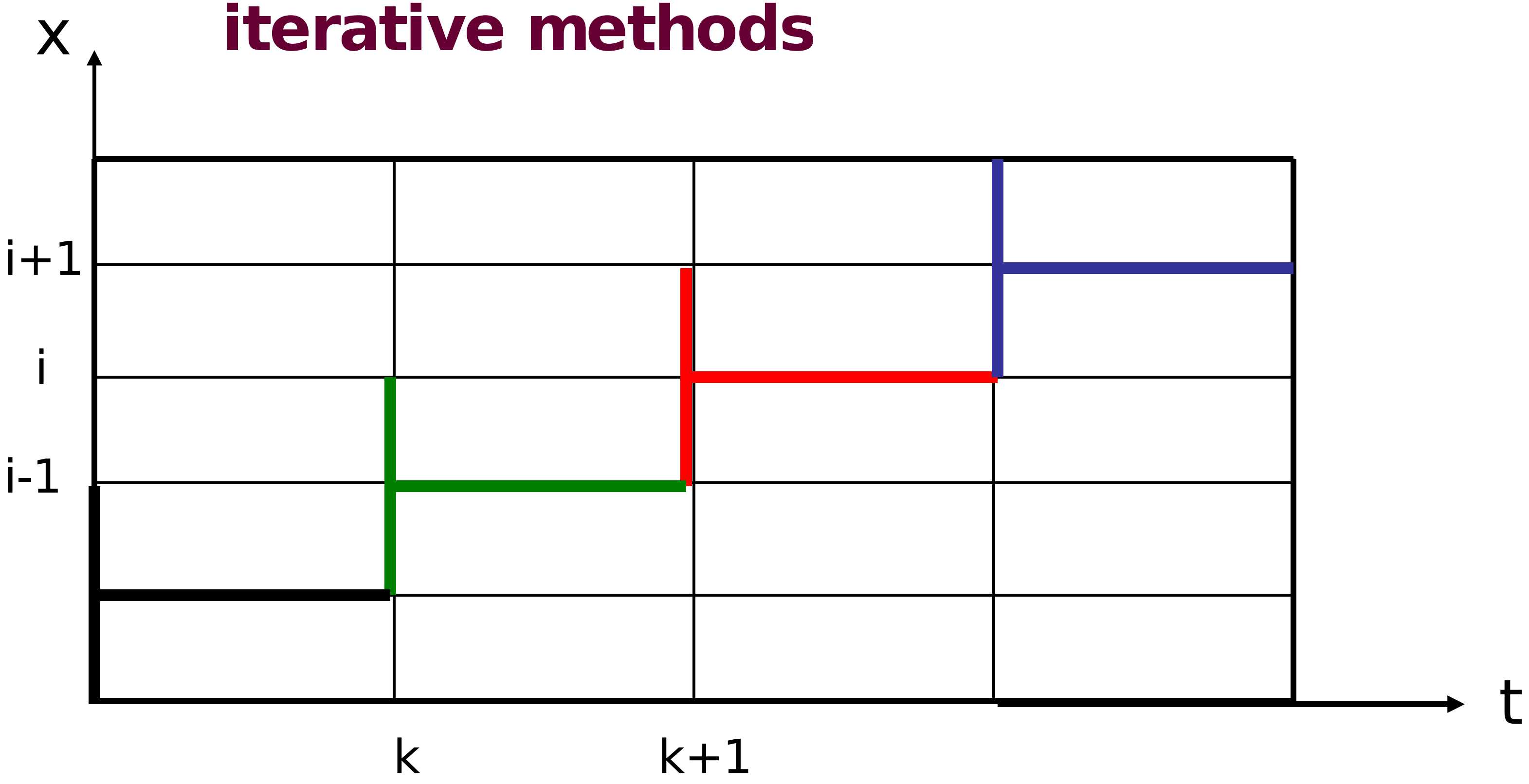
Synchronous updating

Jacobi and JOR iterative methods



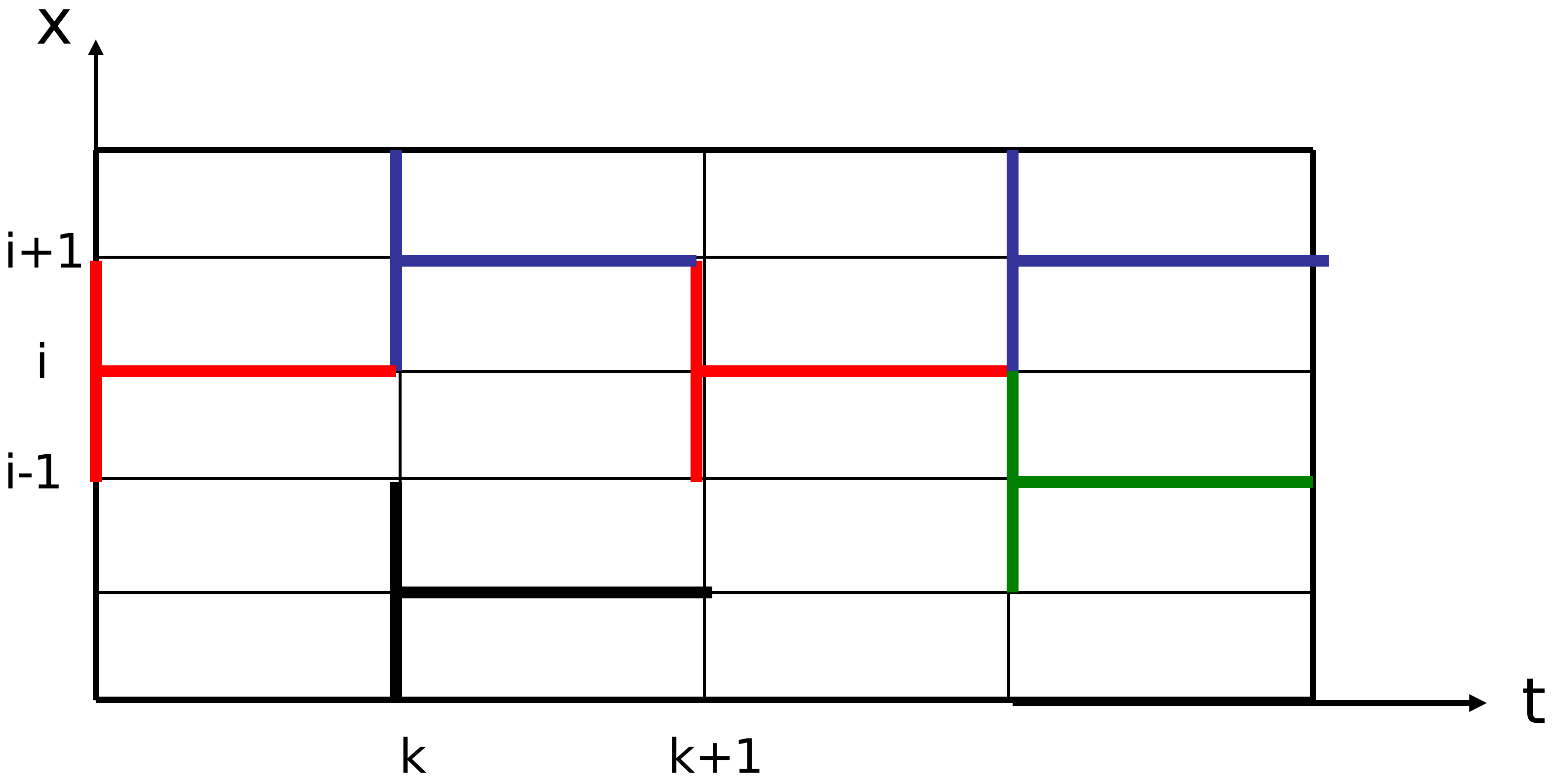
Sequential updating

Gauss-Seidel and SOR iterative methods



Swarm updating

Partial Random Synchronicity



Swarm updating methods

- Quantized over-relaxation method

$\left|(Au-b)_i\right|<1$	Do nothing
$\left|(Au-b)_i\right|\geq 1$	$u_i^{(k+1)}=u_i^{(k)}-1; u_{i+1}^{(k+1)}=u_{i+1}^{(k)}+1$
$\left|(Au-b)_i\right|\leq 1$	$u_i^{(k+1)}=u_i^{(k)}+1; u_{i+1}^{(k+1)}=u_{i+1}^{(k)}-1$

- Gradient method

$$u_i^{(k+1)}=u_i^{(k)}-\varepsilon\,\mathrm{sign}\left(\frac{\partial E}{\partial u_i^{(k)}}\right)$$

$$E=\sum_i (Au-b)_i^2$$

Quantized over-relaxation

- +converges in many cases
- -proof restricts norm of A
- -proof is only a sufficient condition

- -Only 1-dim cases (practically) solvable

Gradient Method

- +Converges

- -Needs one level more neighbors than interacting units
- -slow

Modified iterative scheme

- Iterative scheme for solving $Au = b$

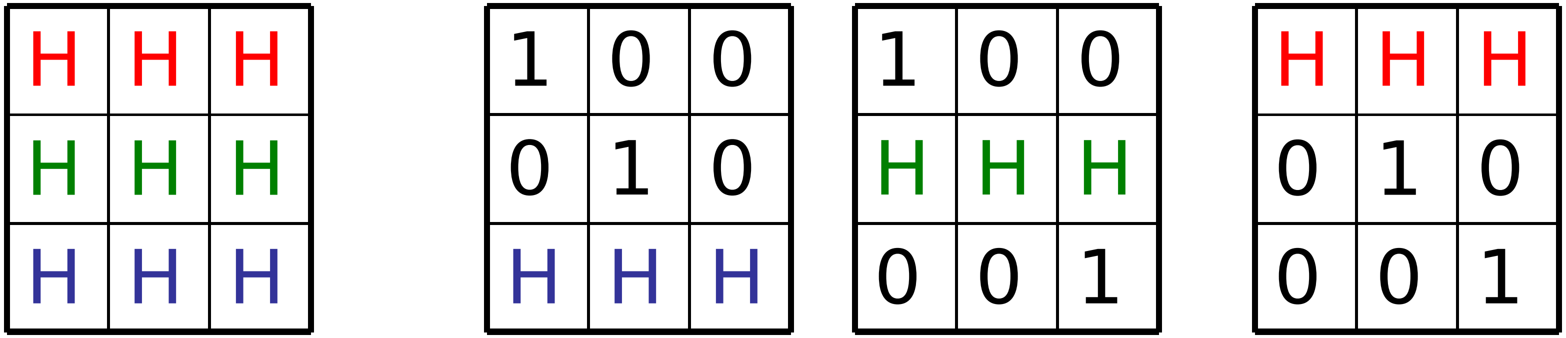
$$u^{(k+1)} = Hu^{(k)} + W^{-1}b$$

$$H = I - W^{-1}A$$

- Modified iterative scheme

Sequential updating

- Basic Idea: from the synchronous iteration matrix form ‘row matrices’



H	H	H
H	H	H
H	H	H

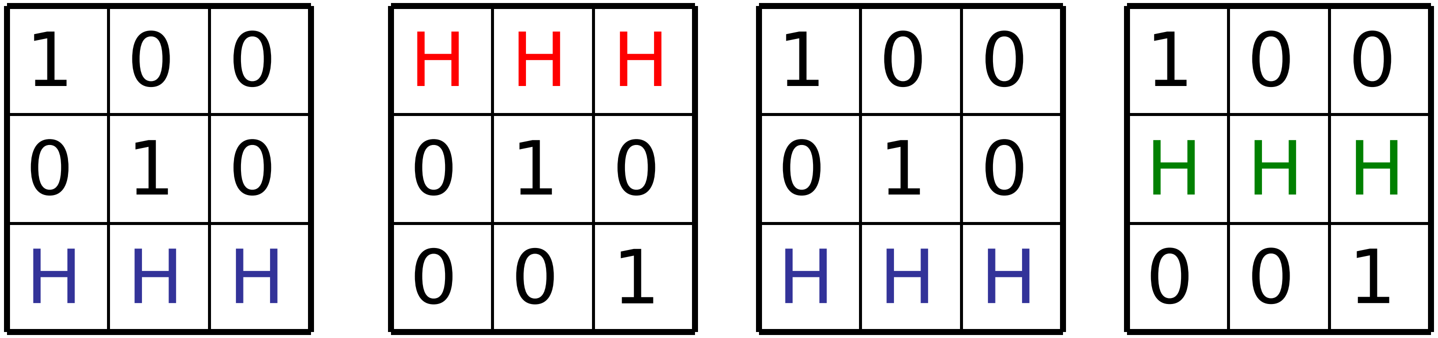
1	0	0
0	1	0
H	H	H

1	0	0
H	H	H
0	0	1

H	H	H
0	1	0
0	0	1

Repetitiveness

Same matrices appear more than once in an updating cycl



1	0	0
0	1	0
H	H	H

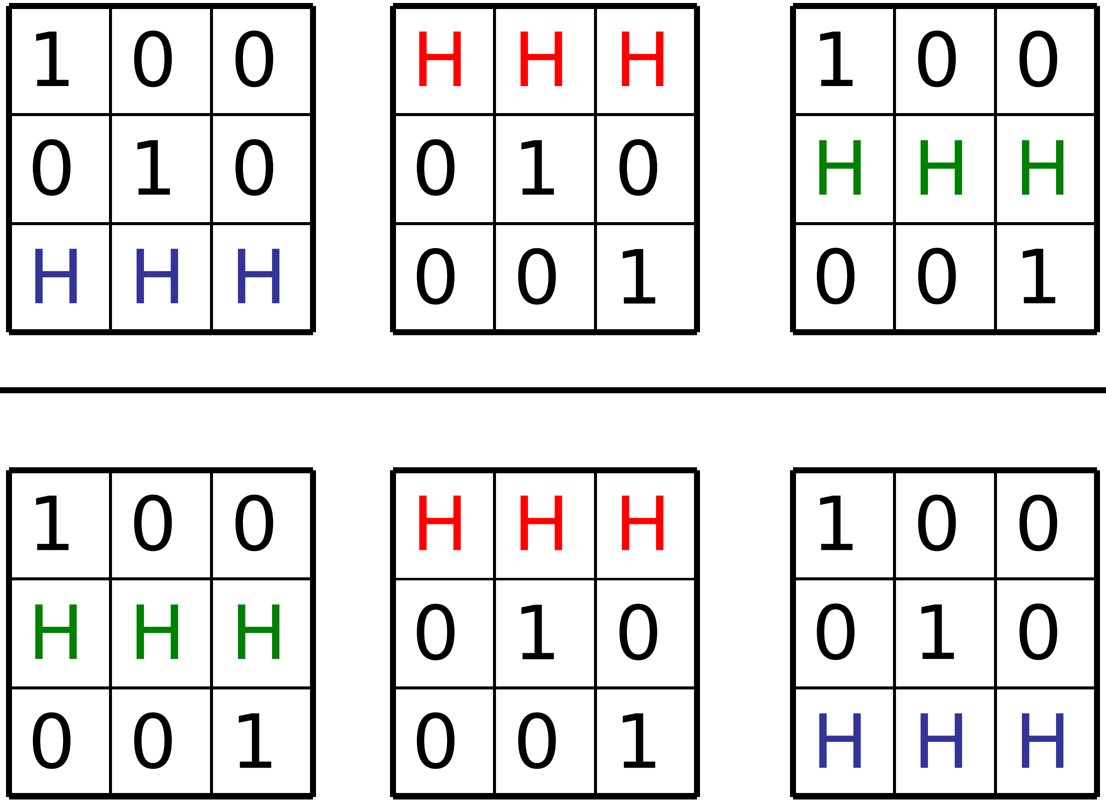
H	H	H
0	1	0
0	0	1

1	0	0
0	1	0
H	H	H

1	0	0
H	H	H
0	0	1

‘Randomness’

Updating cycles are not identical



1	0	0
0	1	0
H	H	H

H	H	H
0	1	0
0	0	1

1	0	0
H	H	H
0	0	1

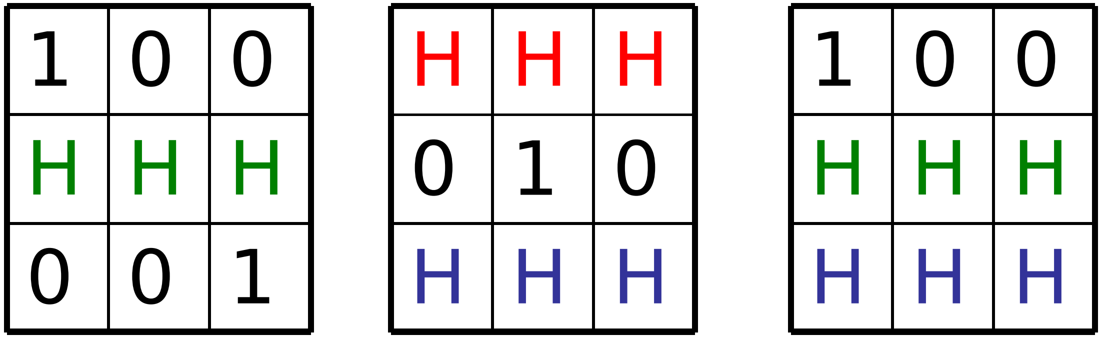
1	0	0
H	H	H
0	0	1

H	H	H
0	1	0
0	0	1

1	0	0
0	1	0
H	H	H

Partial synchronicity

Some matrices update more than one component



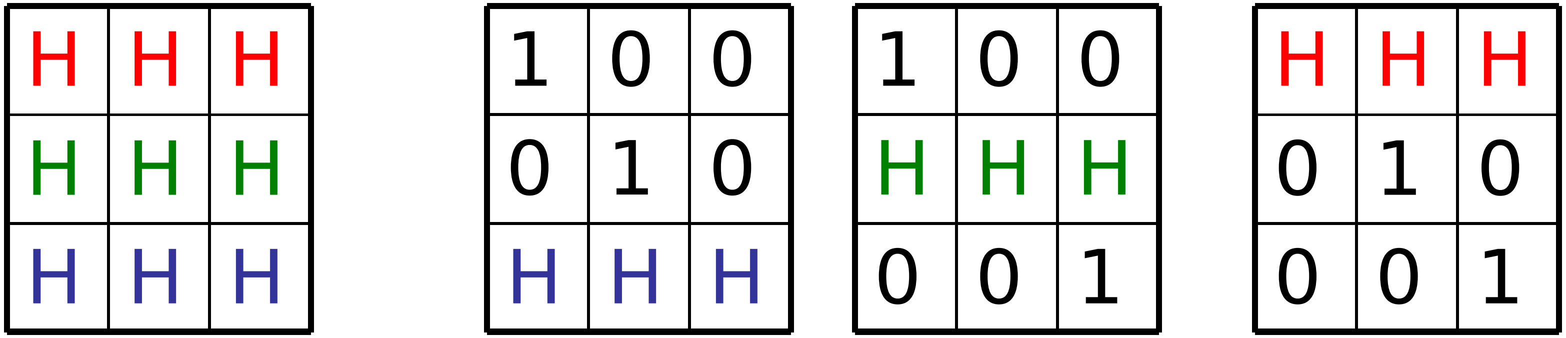
1	0	0
H	H	H
0	0	1

H	H	H
0	1	0
H	H	H

1	0	0
H	H	H
H	H	H

Basic Decomposition

- From synchronous to sequential
  - a special case of asynchronous order



H	H	H
H	H	H
H	H	H

1	0	0
0	1	0
H	H	H

1	0	0
H	H	H
0	0	1

H	H	H
0	1	0
0	0	1

Asynchronous set

Shuffle the matrices and consider all possible permutations

1	0	0
0	1	0
H	H	H

H	H	H
0	1	0
0	0	1

1	0	0
H	H	H
0	0	1

Notation

H = synchronous

H”’ = asynchronous (even)

H”’$_{\text{(odd)}}$ = asynchronous (odd)

Proof of convergence: step 1
Asynchronicity

Case	Spectral radius of H	Spectral radius of H'''	Spectral radius of H_{odd}'''
(i)	<1	<1	<1 or ≥1
(ii)	<1	≥1	
(iii)	≥1	<1	<1 or ≥1
(iv)	≥1	≥1	

Asynchronous conditions for convergence for case (i) and (iii) are not strict

Proof of convergence: step 2a Repetition

- Notation $\begin{array}{|c|c|c|} \hline 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline H & H & H \\ \hline \end{array} \equiv I_{H@3}$

- Insert extra matrix in H''' $I_{H@N}\, I_{H@(N-1)} \ldots\, I_{H@i} \ldots\, I_{H@2}\, I_{H@1}$

ρ (ABCD) = ρ (DABC) = ρ (CDAB)=….= ρ (any cyclic permutation) →

→ $I_{H@N}\, I_{H@(N-1)} \ldots\, I_{H@i} \ldots\, I_{H@2}\, I_{H@1} = I_{H@i}\, H'''$

Proof of convergence: step 2b Repetition

Notation ∞-norm ≡ $\|A\|_\infty \equiv \max_i \sum_{j=1}^{N} |a_{i,j}|$

In general $\|I_{H@i}\|_\infty \geq 1$ Assume(1) $\|I_{H@i}\|_\infty = 1$ →

→ $\|I_{H@i} H_p'''\|_\infty \leq \|I_{H@i}\|_\infty \|H_p'''\|_\infty = \|H_p'''\|_\infty$

In general ρ(H’’’) < 1 $\not\rightarrow$ $\|H_p'''\|_\infty < 1$ Assume (2)

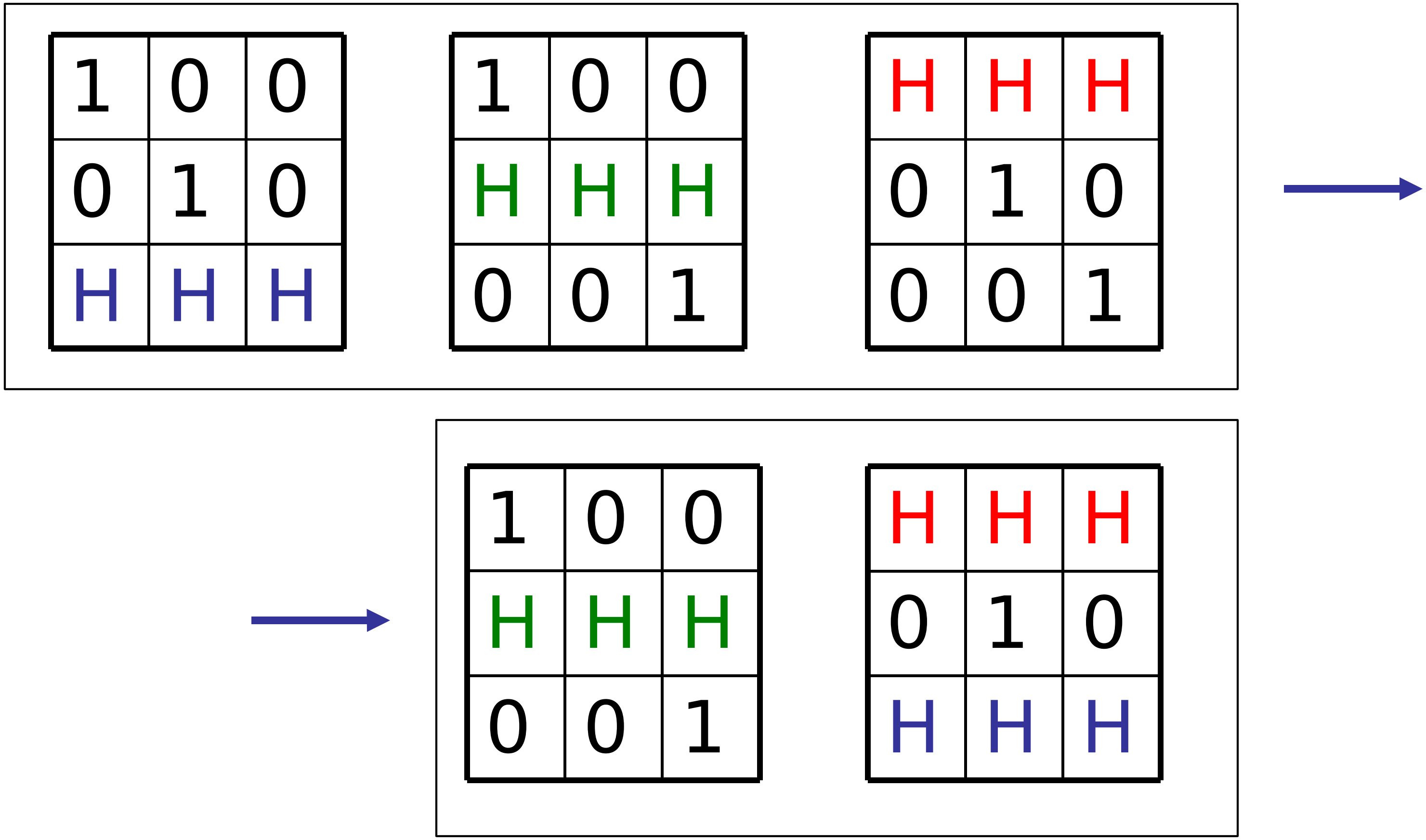
$\boxed{\|I_{H@i} H_p'''\|_\infty < 1}$ → ρ($I_{H@i}$ H_p’’’) < 1

(1) And (2) not strict assumptions:

(1) works e.g. whenever H’’’ has zero diagonal as , e.g., in Jacobi

(2) works if ∞–norm is replaced with L-norm (a suitable L matrix always ∃)

Proof of convergence: step 3a
Partial synchronicity



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ H & H & H \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ H & H & H \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} H & H & H \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 \\ H & H & H \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} H & H & H \\ 0 & 1 & 0 \\ H & H & H \end{bmatrix}$$

Intermediate between H and H'''

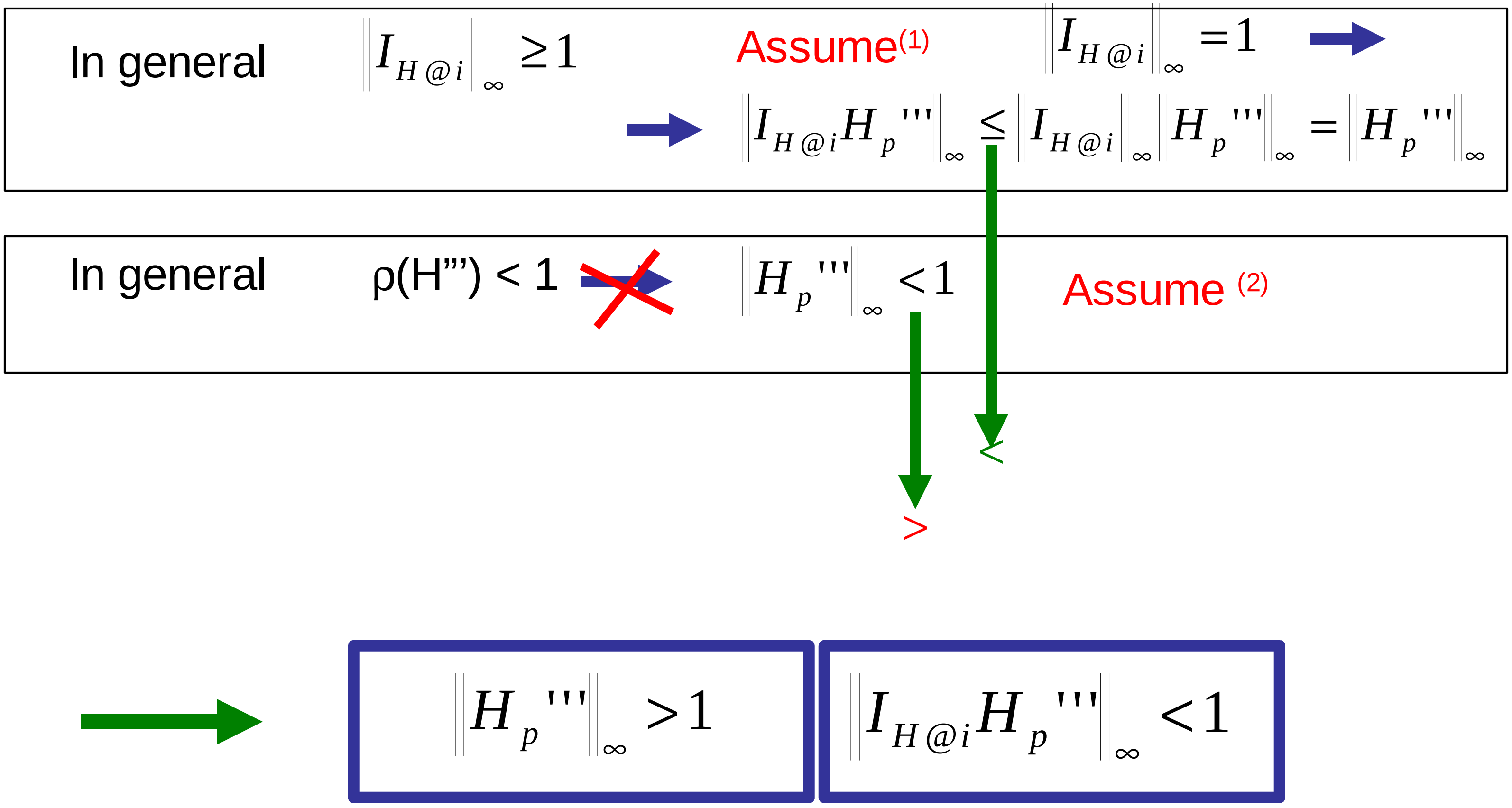
Partial synchronicity
Sync/Async Transition

Case	Spectral radius of H	Spectral radius of H'''	Spectral radius of H_{odd}'''
(i)	<1	<1	<1 or ≥ 1
(ii)	<1	≥ 1	
(iii)	≥ 1	<1	<1 or ≥ 1
(iv)	≥ 1	≥ 1	

Synchronous/Async Transition

Asynchronous conditions for convergence

Repetition may lead to convergence:



In general $\left\|I_{H@i}\right\|_\infty \geq 1$

Assume(1) $\left\|I_{H@i}\right\|_\infty = 1$ →

→ $\left\|I_{H@i}H_p'''\right\|_\infty \leq \left\|I_{H@i}\right\|_\infty \left\|H_p'''\right\|_\infty = \left\|H_p'''\right\|_\infty$

In general ρ(H''') < 1 $\cancel{\Rightarrow}$ $\left\|H_p'''\right\|_\infty < 1$ Assume (2)

<

\>

→ $\boxed{\left\|H_p'''\right\|_\infty > 1}$ $\boxed{\left\|I_{H@i}H_p'''\right\|_\infty < 1}$

Diverges

Converges

Results

- main result: convergence under broad conditions makes pattern engineering by swarms feasible
- Useful result: Rate of convergence ~ log ρ(H)
- Surprising results :

  -Sync/Async Transition

  -Convergence in cases of synchronous and/or asynchronous non convergence

Convergence

- Much broader range of conditions than previous swarm methods:

  (includes, e.g., 2-dim 2nd order diff. eqs.)

- Under this broad range of conditions:
  - Partially Synchronously random-updating linear maps reach same fixed point as their synchronously updating counterparts

Rate of convergence

Swarm	$\sim r$
SOR	$\sim 2r$
JOR	$\sim r$
LQOR	$< r$
Swarm gradient	$<< r$

$r \equiv (\text{number of cycles})^{-1}$

Clocktime vs. rate of convergence

- Average updating time of a unit : t
- Number of units: N
- Average time of an update cycle : T
- JOR (synchronous) : T=t
- SOR (sequential) : T=Nt
- Clock time to converge
- JOR : ~t/r
- SOR : ~Nt/2r

Synchronous/Asynchr. transition

- Swarm may converge even when synchronous updating does not, if the synchronicity is reduced below a critical level.

Reason for *convergence* in cases of synch. and/or asynchr. *non-convergence*

- Some updating orders are <u>inaccessible in synchronous and/or asynchronous</u> cases
- Fluctuations make accessible more updating orders, some of which lead to convergence
- Example:

$$H = \begin{bmatrix} 0 & -0.435 & -0.87 \\ 0.435 & 0 & 0.87 \\ 0.87 & -0.87 & 0 \end{bmatrix}$$

$\rho(H)=1.304$

$\rho(H'''_{p(even)}) = 1.063$

$\rho(H'''_{p(odd)})=2.18$

Effective contraction sequences: $I_{H@3} I_{H@2} I_{H@3}$ and $I_{H@3} I_{H@1} I_{H@3}$

End

Thank you for your attention